Delay-Induced Oscillations and Instability in a Nonlinear Supply Chain Inventory System

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ABSTRACT

This study examines delay-induced oscillations and stability loss in a continuous-time supply chain inventory system. The focus is on a single-echelon inventory model characterized by constant customer demand and an order-up-to policy, subject to a fixed replenishment lead time. The ordering rate is influenced by the deviation of on-hand inventory from the desired target level through linear and cubic feedback terms, resulting in a scalar delay differential equation with nonlinear delayed feedback. By analyzing deviations from the target inventory level, we derive the equilibrium point and assess its local stability by examining the associated characteristic equation. For positive feedback gain, we derive an explicit expression for the critical lead time at which a Hopf bifurcation occurs. Below this threshold, the equilibrium is locally asymptotically stable, and inventory deviation diminishes to zero; when the delay surpasses the critical value, a pair of complex conjugate eigenvalues crosses the imaginary axis, leading to persistent oscillatory adjustments in the system. The linear–cubic feedback structure underscores how stronger corrective reactions to inventory discrepancies, combined with significant lead times, can amplify rather than mitigate fluctuations. These findings elucidate a clear delay–margin relationship for a simple order-up-to policy and provide a dynamic explanation for structurally induced bullwhip behavior: even with constant demand, the interaction between feedback gains and lead time can generate sustained oscillations in inventory levels. Managerial implications regarding the joint selection of feedback parameters and permissible lead times are discussed, and extensions to more realistic multi-echelon settings and stochastic demand models are proposed.

Keywords: Delay differential equations, inventory control, Hopf bifurcation, bullwhip effect, delay-induced oscillations.

INTRODUCTION

Supply chains are inherently dynamic systems characterized by the interaction of material and information flows across multiple echelons, often under conditions of uncertainty and delays. Decisions regarding inventory and production made at one stage can propagate both upstream and downstream, frequently resulting in unintended amplification. A notable manifestation of this phenomenon is the bullwhip effect, typically defined as the amplification of demand variability as orders progress from downstream customers to upstream suppliers. Empirical and analytical studies have demonstrated that bullwhip behavior is associated with excessive inventory, suboptimal capacity utilization, and a decline in customer service performance.[1].

A substantial body of research has employed system dynamics and control-theoretic methodologies to model supply chain behavior. Traditional replenishment strategies, such as the order-up-to (OUT) policy, alongside more comprehensive frameworks like the Automatic Pipeline, Variable Inventory, and Order-Based Production Control System (APVIOBPCS), offer linear feedback mechanisms that integrate demand forecasts, inventory discrepancies, and pipeline modifications. These frameworks have been scrutinized using linear control theory to establish stability conditions and to quantify the bullwhip effect across various parameter configurations and lead times.[2]. Recent advancements in control systems, such as model predictive control (MPC), have been proposed for the coordination of inventories across multiple echelons. These systems demonstrate enhanced tracking performance and a notable reduction in the bullwhip effect when compared to traditional OUT and APVIOBPCS-based policies. [3].

While a significant portion of this research emphasizes stability and variance amplification in predominantly linear models, numerous studies have highlighted that supply chains can also exhibit pronounced nonlinear and even chaotic behavior. System dynamics models of production–distribution structures and multi-echelon supply chains, particularly those with capacity constraints and nonlinear decision rules, have been demonstrated to produce complex dynamics. These include high-order cycles and irregular oscillations, which are indicative of chaos.[4]. In recent studies, a dynamic model of a three-stage maritime supply chain was examined utilizing nonlinear system tools. The analysis revealed chaotic trajectories in inventory and flow variables, and it was demonstrated that appropriate control strategies can mitigate chaos and synchronize the system [5]. These findings indicate that, in addition to variance amplification, supply chain feedback structures have the potential to drive the system into qualitatively distinct behavioral regimes.

Despite these advancements, the body of literature concerning supply chain dynamics remains relatively limited when compared to the extensive research on optimization and static design problems. A recent review of supply chain dynamics, control, and disruption management highlighted that numerous contributions continue to focus on equilibrium-based or steady-state analyses, while rigorous investigations into the dynamic systems of complex behavior are still comparatively infrequent [6]. Specifically, studies that link delay-induced oscillations and stability loss to simple inventory policies within a continuous-time framework are scarce. Most models related to the bullwhip effect either employ discrete-time formulations or depend on higher-dimensional nonlinear structures, where analytical tractability is limited, and chaos is primarily documented through simulation.

In the field of nonlinear dynamics, research on delay differential equations (DDEs) has demonstrated that time delays can significantly alter the qualitative behaviour of systems that are otherwise stable. In ecological and bioeconomic models, the incorporation of discrete or distributed delays in feedback terms can trigger Hopf bifurcations, resulting in sustained oscillations as the delay increases[7]. Similarly, economic and financial models with delayed adjustment mechanisms exhibit analogous phenomena: as the time delay increases, equilibria may lose stability through Hopf bifurcation, leading to the emergence of periodic solutions, and further parameter variations can result in period doubling cascades and chaotic attractors[8]. These findings underscore the role of delay as a critical structural feature that can function as a bifurcation parameter, even in low-dimensional systems.

This observation naturally motivates a re-examination of supply chain dynamics through the lens of delay-differential equations. Lead times in production and transportation, information-processing lags, and decision review periods introduce delays between the detection of inventory imbalances and the implementation of corrective actions via replenishment orders. In numerous analytical models, these delays are either overlooked, approximated by integer time steps in discrete-time formulations, or incorporated without leveraging their role as

explicit bifurcation parameters. By developing continuous-time inventory models with explicit delay terms, it is possible to directly relate supply chain phenomena, such as the bullwhip effect, to established delay-differential equation theory on stability switches and Hopf bifurcations, and in more complex scenarios, even chaotic dynamics. This study focuses on the onset of oscillations and instability.

In this study, we examine a single-echelon supply chain characterized by constant external demand and a nonlinear order-up-to-type inventory policy, which is subject to a fixed replenishment lead time. The ordering rate is influenced by the deviation between the actual inventory and the desired target level, utilizing a combination of linear and cubic feedback terms. This configuration results in a scalar delay differential equation with nonlinear delayed feedback, structurally akin to other low-dimensional DDE models explored in the control and applied dynamics literature. Building on this formulation, we seek to elucidate how the interaction between nonlinear feedback strength and lead time can transition the system from stable inventory adjustment to sustained oscillations and, under more extreme parameter conditions, to a loss of boundedness in the inventory level.

The contributions of this study are threefold. Firstly, we derive the equilibrium of the delayed inventory system and perform a local stability analysis by examining its associated characteristic equation. By considering the delay and linear feedback gain as bifurcation parameters, we establish explicit conditions under which the equilibrium loses stability via a Hopf bifurcation, thereby providing a rigorous characterization of delay-induced oscillations in the inventory level. Secondly, we explore the role of the cubic feedback term in shaping the nonlinear response of the system beyond the Hopf threshold and in amplifying or dampening inventory fluctuations induced by the delay. Thirdly, we interpret these dynamics within the context of supply chain management by relating persistent oscillations and instability to severe manifestations of the bullwhip effect and discussing how overly aggressive adjustment policies, when combined with non-negligible lead times, can compromise system performance. Based on this analysis, we propose managerial guidelines for the selection of feedback parameters that avoid undesirable dynamics and highlight extensions towards multi-echelon and stochastic-demand settings.

LITERATURE REVIEW

Research on supply chain dynamics has shown that production–inventory systems behave as feedback control loops, in which ordering policies, forecasting rules, and lead times jointly determine variability and stability. A large stream of work on the bullwhip effect documents how demand information is amplified upstream and analyzes its operational and behavioral causes, as well as the impact of practices such as order batching on inventory oscillations and service performance [9]. From this dynamic perspective, the IOBPCS/APIOBPCS/APVIOBPCS family of models has been widely used to represent production and inventory control. Using difference-equation and control-theoretic tools, these models have been studied to characterize stability regions, drift elimination, and the conditions under which parameter choices and lead times aggravate or mitigate the bullwhip effect [10].

Beyond linear models, several studies have explored nonlinear and chaotic behaviors in supply chains. Chaotic supply chain systems have been proposed and analyzed using bifurcation diagrams and Lyapunov exponents, revealing transitions from regular cycles to chaos and multistability under deterministic demand [11]. Other studies have formulated low-dimensional nonlinear supply chain models and designed active controllers to synchronize or stabilize chaotic dynamics, explicitly demonstrating that inappropriate feedback gains can drive the system into chaotic regimes and that suitable control laws can suppress this behavior [12].

In parallel, the delay-systems literature has established that time delays can drastically alter the system behavior. Studies on delay differential equations (DDEs) in economics and finance show that introducing a delay in adjustment rules can cause equilibria to lose stability via Hopf bifurcation and lead to periodic, quasi-periodic, and chaotic dynamics as the delay and feedback parameters vary [8]. More general analyses of nonlinear DDEs with constant delay have provided detailed descriptions of stability switches, basins of attraction, and paired Hopf bifurcations, confirming that even low-dimensional delayed systems can exhibit rich dynamical scenarios [13]. However, only a limited number of studies have formulated supply chain inventory policies directly as simple continuous-time DDEs with explicit nonlinear delayed feedback and then linked delay-induced oscillations and instability to familiar phenomena such as the bullwhip effect. This motivated the development of the delayed inventory model in this study.

MODEL FORMULATION
We consider a single-echelon supply chain in which a retailer (or production–inventory facility) serves a constant external demand and places replenishment orders with an upstream supplier. The system is viewed as a feedback control loop: deviations between the actual on-hand inventory and a desired target level trigger ordering decisions, which affect the future inventory position after a fixed lead time. This view is consistent with classical production–inventory control structures, such as order-up-to (OUT) policies and the broader IOBPCS/APVIOBPCS family, where inventory and pipeline feedback are used to regulate orders and mitigate bullwhip.

System assumptions and variables
We make the following simplifying assumptions: First, the supply chain consists of a single stage that faces exogenous and constant customer demand. Then, replenishment orders are delivered after a fixed deterministic lead time. Next, there are no capacity constraints or backlogging; the focus is on the dynamic response of the ordering rule and the inventory level. Finally, the ordering policy is of the order-up-to type with nonlinear feedback from the deviation of the inventory level from a prescribed target.

Let $x(t)$ denote the on-hand inventory level at time $t \geq 0$, and variable $D > 0$ be the constant demand rate, then $u(t)$ be the replenishment rate (ordering/production input) at time t, variable $\tau > 0$ be the replenishment lead time, then x^* be the desired (target) inventory level. The inventory balance equation is written as

$$\frac{dx(t)}{dt} = u(t - \tau) - D \tag{1}$$

which states that the inventory changes according to the delayed replenishment inflow $u(t - \tau)$ minus the constant demand outflow D. This structure follows earlier continuous-time production–inventory models, where delivery delays are explicitly modeled as time lags in the inflow.

Nonlinear order-up-to policy with delayed feedback
Classical OUT policies aim to restore the inventory position to a target level by adjusting the order rate in proportion to the inventory discrepancy. To capture stronger corrective reactions and potential nonlinear effects, we assume that the ordering rate depends on the deviation between the target and the current inventory through both linear and cubic feedback terms, namely

$$u(t) = D + \alpha\big(x^* - x(t)\big) + \beta\big(x^* - x(t)\big)^3 \tag{2}$$

where $\alpha \in \mathbb{R}$ is the linear feedback gain and $\beta \in \mathbb{R}$ scales the cubic nonlinearity. For $\beta = 0$, (2) reduces to a standard linear feedback rule around a base-stock level. For $\beta \neq 0$, the cubic

term introduces a nonlinear response: large deviations from the target are corrected more aggressively than small deviations, which, in the presence of delay, may destabilize the system and lead to complex dynamics.

Substituting (2) into the inventory balance (1) yields

$$\frac{dx(t)}{dt} = \alpha\big(x^* - x(t - \tau)\big) + \beta\big(x^* - x(t - \tau)\big)^3 \qquad (3)$$

Equation (3) is a scalar delay differential equation with a nonlinear delayed feedback. The delay τ represents the lead time between ordering and receipt, while α and β capture the aggressiveness of the inventory correction policy.

Deviation form of the mode

For the analysis, it is convenient to express the system in terms of the deviation of the inventory from its target level. Define

$$y(t) = x(t) - x^* \qquad (4)$$

Since x^* is constant, we have

$$\frac{dy}{dt} = \frac{dx}{dt}.$$

Moreover

$$x^* - x(t - \tau) = -y(t - \tau)$$

Substituting into (3) gives

$$\frac{dy(t)}{dt} = -\alpha y(t - \tau) - \beta y^3(t - \tau) \qquad (5)$$

Equation (5) is the core model studied in this study. It represents a first-order delay differential equation in which the rate of change of the inventory deviation at time t depends nonlinearly on the delayed deviation $y(t - \tau)$. The equilibrium corresponding to perfect inventory tracking is $y^* = 0$, i.e., $x = x^*$. As will be shown in the subsequent sections, the interplay between the lead time τ and the feedback gains α and β can cause this equilibrium to lose stability through Hopf bifurcation and can lead to periodic and chaotic oscillations in the inventory level. If desired, the model can be further non-dimensionalized by rescaling time and state variables, but for managerial interpretability we retain the dimensional form (5) and treat α, β, and τ directly as control parameters.

LOCAL STABILITY AND BIFURCATION ANALYSIS

We now investigate the local stability of the equilibrium of the delayed inventory model and show that increasing the lead time can destabilize the system through a Hopf bifurcation. The analysis follows the standard approach for delay differential equations (DDEs): we linearize the system around the equilibrium, derive a transcendental characteristic equation, and then determine the parameter values at which a pair of complex conjugate eigenvalues crosses the imaginary axis. t

Recall that, in deviation form, the inventory dynamics are given by (5) where α and β are feedback gains and $\tau > 0$ is the replenishment lead time. The equilibrium of interest corresponds to perfect inventory tracking, that is, $y(t) = 0$.

Theorem 1. *Assume $\alpha \neq 0$. Then equation (5) admits a unique real equilibrium $y^* = 0$. The local stability of this equilibrium is determined by the linear delay differential equation*

$$\frac{dy(t)}{dt} = -\alpha y(t - \tau) \qquad (6)$$

whose characteristic equation is

$$\lambda + \alpha e^{-\lambda \tau} = 0 \qquad (7)$$

Proof.

Equilibria of (5) satisfy

$$0 = -\alpha \, y^* - \beta \, (y^*)^3 = -y^*(\alpha + \beta(y^*)^2).$$

Hence either $y^* = 0$ or $\alpha + \beta(y^*)^2 = 0$. For $\alpha \neq 0$ and real y^*, the latter would require $(y^*)^2 = -\alpha/\beta$, which has no real solution when $\alpha > 0$ and β is finite. In any case, the economically relevant equilibrium corresponding to the target inventory level is $y^* = 0$. To analyze local stability, let $y(t) = y^* + z(t)$ with $z(t)$ small. Substituting into (5) and expanding around $y^* = 0$ yields

$$\frac{dz(t)}{dt} = -\alpha \, z(t - \tau) - \beta \, z^3(t - \tau),$$

so the linearized dynamics are governed by

$$\frac{dz(t)}{dt} = -\alpha \, z(t - \tau),$$

which is (6) with z in place of y.

We now look for exponential solutions of (6) in the form $y(t) = e^{\lambda t}$, $\lambda \in \mathbb{C}$. Substitution gives

$$\lambda e^{\lambda t} = -\alpha e^{\lambda(t-\tau)},$$

and dividing by $e^{\lambda t} \neq 0$ leads to

$$\lambda + \alpha e^{-\lambda \tau} = 0,$$

which is (7). The local stability of $y^* = 0$ is therefore determined entirely by the roots λ of (7).

Theorem 4.1 shows that locally, the dynamics around the operating point is only determined by the linear part with delay, while the cubic nonlinearity $-\beta y^3(t - \tau)$ does not affect the linear stability boundary (but will later affect the amplitude and shape of periodic solutions). This approach is in line with the Hopf bifurcation analysis of various logistic, predator-prey, and financial system models with a delay: first, derive the characteristics of the equation, and then look for conditions when the roots move from the negative real part to zero.

Theorem 2. *Suppose $\alpha > 0$. Let*

$$\tau_k = \frac{\frac{\pi}{2} + k\pi}{\alpha}, \quad k = 0, 1, 2, \cdots \tag{8}$$

Then

1. *For each k, the characteristic equation (7) has a pair of purely imaginary roots $\lambda = \pm i\omega_0$ with $\omega_0 = \alpha$ when $\tau = \tau_k$.*
2. *For $0 \leq \tau < \tau_0$, all roots of (7) satisfy $Re \, \lambda < 0$, so the equilibrium $y^* = 0$ of (5) is locally asymptotically stable.*
3. *At $\tau = \tau_0$, the equilibrium $y^* = 0$ undergoes a Hopf bifurcation: a pair of complex conjugate eigenvalues crosses the imaginary axis transversally, and a family of small-amplitude periodic solutions bifurcates from the equilibrium.*

Proof. Step 1: Set $\lambda = i\omega$ with $\omega > 0$ in (7):

$$i\omega + \alpha e^{i\omega\tau} = 0$$

Using $e^{-i\omega\tau} = \cos(\omega\tau) - i\sin(\omega\tau)$, we obtain

$$i\omega + \alpha[\cos(\omega\tau) - i\sin(\omega\tau)] = 0.$$

Equating real and imaginary parts gives the system

$$\alpha \cos(\omega\tau) = 0, \quad \omega - \alpha \sin(\omega\tau) = 0 \tag{9}$$

Assuming $\alpha \neq 0$, the first equation implies

$$\cos(\omega\tau) = 0 \Rightarrow \omega\tau = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}. \tag{10}$$

Hence $\sin(\omega\tau) = \sin\left(\frac{\pi}{2} + k\pi\right) = (-1)^k$. Substituting into the second equation in (9) yields
$$\omega - \alpha(-1)^k = 0 \Rightarrow \omega = \alpha(-1)^k.$$
Because $\omega > 0$ and $\alpha > 0$, we choose $\omega_0 = \alpha$. Combining with (10) gives
$$\tau_k = \frac{\frac{\pi}{2} + k\pi}{\omega_0} = \frac{\frac{\pi}{2} + k\pi}{\alpha}, k = 0,1,2, ...,$$
This proves (1).

Step 2: Stability for small delays. General results on scalar DDEs of the form

$$\dot{y}(t) = -ay(t) - by(t - \tau)$$
show that, when the delay is sufficiently small, all roots of the characteristic equation have negative real parts and the equilibrium is asymptotically stable, while increasing τ can lead to stability switches when purely imaginary roots appear. In our case, when $\tau = 0$, (7) reduces to $\lambda + \alpha = 0$, so $\lambda = -\alpha < 0$. By continuity of the roots with respect to parameters, there exists an interval $0 \leq \tau < \tau_0$ on which all roots remain in the left half-plane. The value $\tau_0 = \frac{\pi}{2\alpha}$ is the smallest delay for which (7) admits purely imaginary roots (by part (1)), so the equilibrium is asymptotically stable for $0 \leq \tau < \tau_0$. This proves part (2).

Step 3: Transversality and Hopf bifurcation at τ_0. We now verify that, at $\tau = \tau_0$, the pair of roots $\lambda(\tau), \lambda(\bar{\tau})$ crosses the imaginary axis with nonzero speed as τ passes through τ_0. Define

$$F(\lambda, \tau) = \lambda + \alpha e^{-\lambda\tau}.$$

Then $F(\lambda, \tau) = 0$ is the characteristic equation. Differentiating implicitly with respect to τ gives
$$F_\lambda(\lambda, \tau) \frac{d\lambda}{d\tau} + F_\tau(\lambda, \tau) = 0,$$
so

$$\frac{d\lambda}{d\tau} = -\frac{F_\tau(\lambda, \tau)}{F_\lambda(\lambda, \tau)}.$$

A direct computation yield
$$F_\lambda = 1 - \alpha\tau e^{-\lambda\tau}, F_\tau = -\alpha\lambda e^{-\lambda\tau}.$$
Hence

$$\frac{d\lambda}{d\tau} = \frac{\alpha\lambda e^{-\lambda\tau}}{1 - \alpha\tau e^{-\lambda\tau}}. \tag{11}$$

At the critical point $\tau = \tau_0$, we have $\lambda = i\omega_0 = i\alpha$ and $\omega_0\tau_0 = \frac{\pi}{2}$, so
$$e^{-\lambda\tau_0} = e^{-i\alpha\tau_0} = e^{-\frac{i\pi}{2}} = -i.$$
Thus

$$\alpha\lambda e^{-\lambda\tau_0} = \alpha(i\alpha)(-i) = \alpha^2,$$
and

$$1 - \alpha\tau_0 e^{-\lambda\tau_0} = 1 - \alpha\left(\frac{\pi}{2\alpha}\right)(-i) = 1 + \frac{\pi}{2}i.$$

Substituting into (11) gives

$$\left. \frac{d\lambda}{d\tau} \right|_{\tau=\tau_0} = \frac{\alpha^2}{1 + \frac{\pi}{2} i}.$$

Since $\frac{1}{1+(\pi/2)i}$ has strictly positive real part, we obtain

$$\mathrm{Re}\left(\left. \frac{d\lambda}{d\tau} \right|_{\tau=\tau_0} \right) \neq 0.$$

Therefore, the pair of eigenvalues crosses the imaginary axis transversally as τ passes through τ_0. By the Hopf bifurcation theorem for functional differential equations, a family of small-amplitude periodic solutions bifurcates from the equilibrium $y^* = 0$ at $\tau = \tau_0$, establishing part (3).

Theorem 4.2 provides a clear picture of how the lead time τ affects the inventory dynamics. As long as τ is below $\tau_0 = \frac{\pi}{2\alpha}$, all characteristic roots are to the left of the imaginary axis, so the inventory deviation $y(t)$ returns to zero exponentially; from a managerial perspective, the system is stable and does not induce long-term oscillations. When τ passes τ_0, the pair of complex roots move to the right along the imaginary axis, triggering a Hopf bifurcation and the appearance of small periodic oscillations in the inventory level. For larger values of τ (or for an enlarged α), other roots may shift further, opening up the possibility of more complex dynamical scenarios, as reported in the general DDE literature. In our numerical experiments for economically reasonable parameter ranges, we observed a transition from stable convergence to small-amplitude oscillations and, eventually, to divergence rather than sustained chaotic attractors.

DISCUSSION

The delay-based formulation of the inventory system developed in this study illustrates how even a simple order-up-to-type policy can generate rich dynamic behavior once explicit lead times are considered. At the core of the analysis is a scalar delay differential equation in which the rate of change of the inventory deviation depends on a delayed linear–cubic feedback term, as follows: Rewriting the model in deviation form and linearizing it around the target inventory level yields a characteristic equation whose roots can be studied explicitly, leading to a clear condition for local stability and the onset of oscillations via a Hopf bifurcation.

From a control-theoretic standpoint, the main insight is the explicit trade-off between feedback aggressiveness and the lead time. The feedback gain α captures how strongly the ordering policy reacts to inventory discrepancies, while the delay τ represents the time lag between placing and receiving replenishment orders. The existence of a critical delay $\tau_0 = \frac{\pi}{2\alpha}$ at which a Hopf bifurcation occurs shows that, for a fixed gain, there is a maximum lead time that preserves local stability, and conversely, for a fixed delay, there is an upper bound on the feedback gain beyond which oscillatory responses arise. This "delay–margin" relation is fully transparent in the present delay differential framework and complements previous analyses of APIOBPCS-type models based on discrete-time or purely linear formulations.

Although the cubic feedback term does not affect the linear stability boundary, it plays an important conceptual role in shaping the nonlinear response of the system beyond the Hopf threshold. The presence of a cubic term means that large deviations from the target inventory are penalized more strongly than small deviations, reflecting a managerial tendency to react more forcefully to severe stock imbalances. In the absence of a delay, such a nonlinear correction could be expected to improve the adjustment by pushing the system back more rapidly towards the target. However, in the delayed setting, the same mechanism interacts with the lead time to create the potential for overshooting and sustained oscillations. Thus, the

analysis highlights a subtle point: stronger corrective rules do not necessarily improve dynamic performance when information and material flows are delayed.

These results provide a structural explanation for bullwhip-type phenomena in a highly simplified environment. Even though customer demand is constant, the delayed feedback loop can cause the inventory to adjust oscillatory once the Hopf threshold is crossed. In other words, persistent fluctuations can arise endogenously from the combination of lead time and feedback aggressiveness without any demand variability or exogenous shocks. This perspective complements more traditional explanations of the bullwhip effect based on stochastic demand, forecasting errors, or behavioral biases by showing that the internal control structure alone may be sufficient to generate oscillatory behavior if delays are not properly accounted for in the policy design.

Simultaneously, the scope of the present model must be interpreted with caution. The analysis is local; it characterizes the stability of the equilibrium and the emergence of oscillations in its vicinity, but it does not fully resolve the global dynamics of the nonlinear delayed system. In particular, the long-term boundedness of solutions and the detailed nature of trajectories far from the target level remain unanswered. The broader literature on nonlinear delay differential equations indicates that more complex behaviour, including large-amplitude oscillations and, in some models, even chaotic attractors, may occur when the parameters are varied further. A complete understanding of how such phenomena might arise in inventory contexts would require combining the local Hopf analysis presented here with more advanced global tools and a systematic numerical exploration.

However, from a managerial perspective, the local results already carry a clear message. When designing order-up-to–type policies in environments with non-negligible lead times, it is not sufficient to choose feedback parameters based solely on steady-state considerations or intuition about "fast reaction." The explicit Hopf condition shows that reacting too aggressively in the presence of delay can undermine stability and create structurally induced oscillations, even in the simplest single-echelon setting with a constant demand. This underscores the value of incorporating delay-aware dynamical analysis into supply chain decision-making and suggests that relatively simple analytical models can provide useful guidance on the safe operating region of feedback-based inventory-policy.

CONCLUSSION

This study examined a continuous-time inventory system with a nonlinear order-up-to-type policy and an explicit replenishment delay, modelled as a scalar delay differential equation with linear–cubic delayed feedback. By rewriting the dynamics in the deviation form and linearizing around the target inventory level, we derived a simple characteristic equation whose roots determine the local behavior of the system. For positive feedback gain α, we obtained an explicit expression for the critical delay $\frac{\pi}{2\alpha}$ at which a Hopf bifurcation occurs. Below this threshold, the equilibrium corresponding to the desired inventory level is locally asymptotically stable; exceeding this threshold causes a pair of complex conjugate eigenvalues to cross the imaginary axis and generates oscillatory responses in the inventory level.

From a control perspective, the analysis explicitly shows how lead time and feedback aggressiveness interact in this simple supply chain setting. The Hopf condition shows that for a given replenishment delay, there is an upper bound on the feedback gain that preserves local stability, and conversely, for a given gain, there is a maximum tolerable delay. Thus, the delay differential formulation provides a transparent delay–margin relation for an order-up-to-type policy and clarifies how delay-induced oscillations can arise even under constant demand. These oscillations may be interpreted as a structural manifestation of the bullwhip effect: inventory levels do not converge smoothly to the target but continue to fluctuate owing to the internal dynamics of the feedback loop.

This study focuses on local stability and the onset of oscillations via Hopf bifurcation. We do not attempt a full global analysis of the nonlinear delayed system, and issues such as the long-term boundedness of solutions or the possible occurrence of more complex dynamics remain unresolved. From a managerial standpoint, however, the explicit Hopf threshold already provides a practical guideline: when designing order-up-to-type policies in the presence of non-negligible lead times, feedback gains should be chosen such that the operating point remains in the locally stable region.

Future research can extend this framework in several directions. On the modelling side, incorporating capacity limits, backlog or lost sales, and more realistic saturation effects in the ordering rule would allow a better representation of real supply chains and a more meaningful interpretation of instability. On the analytical side, combining local Hopf analysis with global techniques and numerical exploration would help map the full range of dynamic regimes beyond the bifurcation threshold. Finally, extending the approach to multi-echelon networks and stochastic demand processes would clarify how delay-induced oscillations and instability interact with other sources of variability and risks in supply chain operations.

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