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Super $(a, d) - C_m$ –Antimagic Total Labeling of Generalized Book Graphs

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ABSTRACT

A graph G is considered to admit a C_m –covering if every edge of G lies in a subgraph isomorphic to C_m. We refer to G as super $(a, d) - C_m$ –antimagic total when there exists a bijection $\sigma : V(G) \cup E(G) \rightarrow \{1,2, \dots, |V(G)| + |E(G)|\}$ with $\sigma\big(V(G)\big) = \{1,2, \dots, |V(G)|\}$ such that the weights of all subgraphs C'_m isomorphic to C_m, defined by $w(C'_m) = \sum\{v \in V(C'_m)\}\sigma(v) + \sum\{e \in E(C'_m)\}\sigma(e)$, form an arithmetic sequence $a, a + d, a + 2d, \dots, a + (n-1)d$ for integers $a > 0$ and $d \geq 0$, where n is the number of such subgraphs. In this work, we establish super $(a, d) - C_m$ –antimagic total labelings for the classical book graph and for a generalized book graph constructed by extending the cycle-based structure of the standard model. We further show that the classical book graph appears as a special case of this generalized structure and that both graphs satisfy the super $(a, d) - C_m$ –antimagic total property.

Keywords: Super (a, d) –antimagic total labeling, C_m –antimagic, Book graph, generalized book graph, Graph coverings

INTRODUCTION

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ and $H = (V(H), E(H))$ be finite, simple, and undirected graphs (that is, graphs without loops or parallel edges). Consider a collection $\mathcal{H} = \{H_1, H_2, \dots, H_k\}$ consisting of distinct subgraphs $H_i, 1 \leq i \leq k$, where each H_i is isomorphic to a fixed graph H. If every edge of $\mathcal{G}$ appears in at least one member of this family, then we say that $\mathcal{G}$ admits an H –covering.

A bijection $\gamma: V(\mathcal{G}) \cup E(\mathcal{G}) \rightarrow \{1,2, \dots, | V(\mathcal{G}) | + | E(\mathcal{G}) |\}$ is called an $(a, d) - H$ –antimagic total labeling when the induced weights of all subgraphs H' isomorphic to H,

$$w(H') = \sum_{v \in V(H')} \gamma(v) + \sum_{e \in E(H')} \gamma(e),$$

form an arithmetic progression $a, a + d, \dots, a + (n-1)d$, where $a, d > 0$ and n denotes the number of such subgraphs. If the vertex labels additionally satisfy $f: V(\mathcal{G}) \rightarrow \{1,2, \dots, | V(\mathcal{G}) |\}$, then the labeling is referred to as a super $(a, d) - H$ –antimagic total labeling. This notion was first proposed by Inayah et al. in [1]. Since its introduction, various results concerning this invariant have appeared in the literature. For instance, Inayah et al. in [2] investigated super $(a, d) - H$ –antimagic labelings of certain connected graphs constructed from k isomorphic copies of a given graph, while Baca et al. [3] explored the property for disjoint unions of graphs. Much of the existing work on H –antimagic labelings centers on cycle-antimagic graphs, as discussed in [4], [5], [6], [7], [8], [9], [10], [11], [12].

L. Ratnasasari et al. in [13] conducted a study on the total edge irregularity strength of book graphs and double book graphs, while K.M. Kathiresan et al. [14] investigated C_m −supermagic labelings of various graphs, including one form of generalized book graphs. In the present work, we examine super $(a, d) - C_m$ −antimagic total labelings on book graphs as defined by L. Ratnasasari et al., as well as on a generalization of book graphs based on a definition that slightly differs from the one proposed by K.M. Kathiresan et al. Under our definition, book graphs become a special case of the generalized book graphs. X. Ma et al. [15] stated that stacked book graphs are also cycle-antimagic; however, the definition of the generalized graph introduced in this article is not isomorphic to the stacked book graphs they described. We then show that the generalized book graph admits a super $(a, d) - C_m$ −antimagic total labeling.

RESULT

In this section, we present our main findings concerning super $(a, d) - C_m$ −antimagic total labelings on book graphs and on their generalized counterparts. To establish these results, we begin by formalizing the structure of the book graph that serves as the fundamental object of study. This graph is constructed from multiple copies of a cycle C_m arranged in a prescribed manner. For clarity and completeness, the definition of a book graph is given as follows.

Definition 2.1 The book graph, written $B_m^{(n)}$, is defined as the graph obtained by merging n copies of the cycle C_m along one common edge. The sets of vertices and edges are described as follows

$$V(B_m^{(n)}) = \{c_1, c_2\} \cup \{u_i^j, 1 \le i \le m-2, 1 \le j \le n\}$$

$$E(B_m^{(n)}) = \{c_1c_2\} \cup \{c_1u_1^j, 1 \le j \le n\} \cup \{c_2u_{m-2}^j, 1 \le j \le n\} \cup \{u_i^j u_{i+1}^j, 1 \le i \le m-3, 1 \le j \le n\}$$

Based on this definition, the number of vertices of the book graph $B_m^{(n)}$ is $2 + (m-2)n$, and the number of edges is $1 + (m-1)n$

To facilitate understanding of the definition of a book graph, we provide a representation of the book graph $B_6^{(n)}$ in Figure 1.

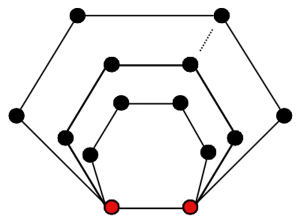

Figure 1. The book graph B_6^n

The red vertices are the common vertices shared by all cycles C_m.

Lemma 2.2 For any $n \ge 2$and $m \ge 3$, the book graph $B_m^{(n)}$is a super $(a, 1)$-C_m-antimagic total graph.

Proof. Let $\sigma: V\left(B_m^{(n)}\right) \cup E\left(B_m^{(n)}\right) \to \{1,2, \dots, (2m-3)n+3)\}$ be a bijective function where

$\sigma(c_i)$	$= i,$	For $i = 1,2$
$\sigma(u_i^j)$	$= 2 + (i-1)n + j,$	For odd $1 \le i \le m-2,$ and for $1 \le j \le n$
$\sigma(u_i^j)$	$= 2 + in - (j-1)$	For even $1 \le i \le m-2,$ and for $1 \le j \le n$
$\sigma(c_1u_1^j)$	$= 2 + (m-2)n + j,$	For $1 \le j \le n$
$\sigma(c_2u_{m-2}^j)$	$= 2 + (2m-5)n + 2n - (j-1),$	For $1 \le j \le n$ and m odd
$\sigma(c_2u_{m-2}^j)$	$= 2 + (2m-5)n + n + j$	For $1 \le j \le n$ and m even
$\sigma(u_i^j u_{i+1}^j)$	$= 2 + (m-1)n + in - (j-1),$	For odd $1 \le i \le m-3,$ and for $1 \le j \le n$
$\sigma(u_i^j u_{i+1}^j)$	$= 2 + (m-1)n + (i-n) + j,$	For even $1 \le i \le m-3,$ and for $1 \le j \le n$
$\sigma(c_1c_2)$	$= (2m-3)n + 3.$	

Note that there are n subgraphs isomorphic to C_m, denoted by $C_m^{(j)}$ for $1 \le j \le n$, where the sets of vertices and edges are described as follow

$$
\begin{aligned}
V\left(C_m^{(j)}\right) &= \{c_1, c_2\} \cup \{u_i^j, 1 \le i \le m-2\} \\
E\left(C_m^{(j)}\right) &= \{c_1 c_2\} \cup \{c_1 u_1^j\} \cup \{c_2 u_{m-2}^j\} \\
&\quad \cup \{u_i^j u_{i+1}^j, 1 \le i \le m-3\}
\end{aligned}
$$

for $1 \le j \le n$. It can be observed that every edge in $B_m^{(n)}$ belongs to at least one $C_m^{(j)}$ subgraph, then $B_m^{(n)}$ admits an C_m −covering.

Next, the weight of each subgraph $C_m^{(j)}$ will be determined by considering two cases for m, namely when m is odd and when m is even.

Case I. For odd m, the sum of the vertex labels of the subgraph $C_m^{(j)}$ is given as follows,

$$
\begin{aligned}
\textstyle\sum_{u\in V\left(C_m^{(j)}\right)} \sigma(u) &= 1 + 2 + \textstyle\sum_{i=1,\ i \text{ odd}}^{m-2} [2 + (i-1)n + j] + \textstyle\sum_{i=2,\ i \text{ even}}^{m-3} [2 + in - (j-1)] \\
&= 3 + \left(\tfrac{m-1}{4}\right)(2(2+j) + n(m-3))\left(\tfrac{m-3}{2}\right)(3-j) + n\left(\tfrac{m-3}{2}\right)\left(\tfrac{m-1}{2}\right) \\
&= \tfrac{1}{2}m^2 n - 2mn + \tfrac{5}{2}m + \tfrac{3}{2}n - \tfrac{5}{2} + j
\end{aligned}
$$

and the sum of the edge labels is given as follows

$$
\begin{aligned}
\textstyle\sum_{uv\in E\left(C_m^{(j)}\right)} \sigma(uv) &= [2 + (m-2)n + j] + [2 + (2m-5)n + 2n - (j-1)] \\
&\quad + \textstyle\sum_{i=1,\ i \text{ odd}}^{m-4} [2 + (m-1)n + in - (j-1)] \\
&\quad + \textstyle\sum_{i=2,\ i \text{ even}}^{m-3} [2 + (m-1)n + (i-1)n + j] + [(2m-3)n + 3] \\
&= [2 + (m-2)n + j] + [2 + (2m-5)n + 2n - (j-1)] \\
&\quad \left(\tfrac{m-3}{4}\right)\big(2(3-j) + n(3m-5)\big) + \left(\tfrac{m-3}{4}\right)\big(2(2+j) + n(3m-5)\big) \\
&\quad + [(2m-3)n + 3] \\
&= \tfrac{3}{2}m^2 n - 2mn - \tfrac{1}{2}n + \tfrac{5}{2}m + \tfrac{1}{2}
\end{aligned}
$$

Summing these two quantities yields the weight of the subgraph $C_m^{(j)}$, as shown below

$$
w\left(C_m^{(j)}\right) = n(2m^2 - 4m + 1) + 5m - 2 + j
$$

for $1 \le j \le n$.

Case II. For even m, the sum of the vertex labels of the subgraph $C_m^{(j)}$ is given as follows

$$
\begin{aligned}
\textstyle\sum_{u\in V\left(C_m^{(j)}\right)} \sigma(u) &= 1 + 2 + \textstyle\sum_{i=1,\ i \text{ odd}}^{m-3} [2 + (i-1)n + j] \\
&\quad + \textstyle\sum_{i=2,\ i \text{ even}}^{m-2} [2 + in - (j-1)] \\
&= 3 + \left(\tfrac{m-2}{4}\right)(2(2+j) + n(m-4))\left(\tfrac{m-2}{4}\right)(2(3-j) + nm). \\
&= \tfrac{1}{2}m^2 n - 2mn + \tfrac{5}{2}m + 2n - 2
\end{aligned}
$$

and the sum of the edge labels is given as follows

$$\begin{aligned}
\sum_{uv\in E\left(C_m^{(j)}\right)} \sigma(uv) = \; & [2 + (m-2)n + j] + [2 + (2m-5)n + n + j] \\
& + \sum_{i=1,\ i\,\text{odd}}^{m-3} [2 + (m-1)n + in - (j-1)] \\
& + \sum_{i=2,\ i\,\text{even}}^{m-4} [2 + (m-1)n + (i-1)n + j] \; + \; [(2m-3)n + 3] \\
= \; & [2 + (m-2)n + j] + [2 + (2m-5)n + n + j] \\
& + \left(\tfrac{m-2}{4}\right)\big(2(3-j) + n(3m-4)\big) + \left(\tfrac{m-4}{4}\right)\big(2(2+j) + 3n(m-2)\big) \\
& + [(2m-3)n + 3] \\
= \; & \tfrac{3}{2}m^2 n - 2mn + \tfrac{5}{2}m - n + j
\end{aligned}$$

By combining the two sums, we obtain the weight of the subgraph $C_m^{(j)}$ as follows

$$w\left(C_m^{(j)}\right) = n(2m^2 - 4m + 1) + 5m - 2 + j.$$

Both the even and odd cases of m yield the same weight for the subgraph $C_m^{(j)}$, namely $n(2m^2 - 4m + 1) + 5m - 2 + j$ for $1 \leq j \leq n$. From the weights obtained for each j, it can be seen that these weights form an arithmetic progression with common difference 1. Based on this fact, and noting that the smallest label among the $2 + (m-2)n$ labels is a vertex label, it follows that the book graph $B_m^{(n)}$ is a super $(a, 1)$-antimagic total graph with $a = n(2m^2 - 4m + 1) + 5m - 1$. □

In Figure 2, we present a labeling of the book graph $B_4^{(3)}$ that realizes a super (70,1)-antimagic total labelling

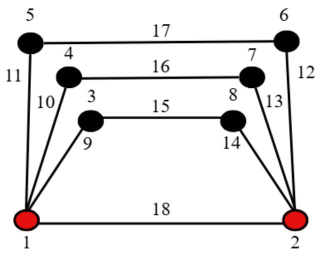

Figure 2. The graph $B_4^{(3)}$ with super $(70, 1)$ −antimagic total labeling

Before developing our results, it is helpful to introduce a more general structure that still keeps the main idea of book graphs but allows them to be formed in a more flexible way. We proceed to generalize the notion of a book graph, introducing what we term a generalized book graph, whose definition is given below.

Definition 2.3 A generalized book graph is defined as the graph formed from $n$copies of the cycle graph $C_m$that share r vertices and $r - 1$ edges in common, where $r < m$. We denote this graph by $B_{r,m}^{(n)}$. The sets of vertices and edges are described as follows

$$\begin{aligned}
V\left(B_m^{(n)}\right) \; = \; & \{c_k, 1 \leq k \leq r\} \\
& \cup \left\{u_i^j, 1 \leq i \leq m - r, 1 \leq j \leq n\right\} \\
E\left(B_m^{(n)}\right) \; = \; & \{c_k c_{k+1}, 1 \leq k \leq r - 1\} \\
& \cup \left\{c_1 u_1^j, 1 \leq j \leq n\right\} \cup \left\{c_r u_{m-r}^j, 1 \leq j \leq n\right\} \\
& \cup \left\{u_i^j u_{i+1}^j, 1 \leq i \leq m - (r-1), 1 \leq j \leq n\right\}.
\end{aligned}$$

Thus, it follows that the generalized book graph $B_{r,m}^{(n)}$ has $r + (m - r)n$ vertices and $(r - 1) + (m - r + 1)n$ edges. Figure 3 presents a representation of the generalized book graph $B_{4,7}^{(2)}$.

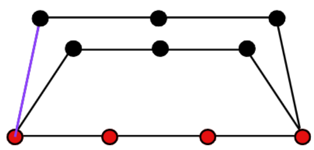

Figure 3. The generalized book graph$B_{4,7}^{(2)}$

Note that the book graph previously discussed can be recognized as a special instance of the generalized book graph corresponding to $r = 2$. Consequently, the labeling function introduced in the proof of Lemma 2.2 can be generalized to the following form,

$$
\begin{array}{rll}
\sigma(c_k) & = k, & \text{For } 1 \le k \le r \\
\sigma(u_i^j) & = r + (i-1)n + j, & \text{For odd } 1 \le i \le m-r, \text{ and for } 1 \le j \le n \\
\sigma(u_i^j) & = r + in - (j-1) & \text{For even } 1 \le i \le m-r, \text{ and for } 1 \le j \le n \\
\sigma(c_1 u_1^j) & = r + (m-r)n + j, & \text{For } 1 \le j \le n \\
\sigma(c_r u_{m-r}^j) & = r + (2(m-r)+1)n - (j-1), & \text{For } 1 \le j \le n \text{ and } m-r \text{ odd} \\
\sigma(c_r u_{m-r}^j) & = r + 2(m-r)n + j & \text{For } 1 \le j \le n \text{ and } m-r \text{ even} \\
\sigma(u_i^j u_{i+1}^j) & = r + (m-r+1)n + in - (j-1), & \text{For odd } 1 \le i \le (m-r)-1, \text{ and for } 1 \le j \le n \\
\sigma(u_i^j u_{i+1}^j) & = r + (m-r+1)n + (i-1)n + j, & \text{For even } 1 \le i \le (m-r)-1, \text{ and for } 1 \le j \le n \\
\sigma(c_k c_{k+1}) & = r + (2(m-r)+1)n + k, & \text{For } 1 \le k \le r-1
\end{array}
$$

Similar to the book graph, the generalized book graph contains n subgraphs isomorphic to C_m, denoted by $C_m^{(j)}$ for $1 \le j \le n$. Every edge of $B_{r,m}^{(n)}$ lies in at least one of these subgraphs, indicating that the generalized book graph admits a C_m-covering. The vertex and edge sets of each subgraph $C_m^{(j)}$, for $1 \le j \le n$, are given below

$$
\begin{array}{rcl}
V\left(C_m^{(j)}\right) & = & \{c_k, 1 \le k \le r\} \cup \{u_i^j, 1 \le i \le m-r\} \\
E\left(C_m^{(j)}\right) & = & \{c_k c_{k+1}, 1 \le k \le r-1\} \\
& & \cup \{c_1 u_1^j\} \cup \{c_r u_{m-r}^j\} \\
& & \cup \{u_i^j u_{i+1}^j, 1 \le i \le m-(r-1)\}.
\end{array}
$$

In determining the weight of each subgraph, we again distinguish two cases, when $m - r$ is odd and when $m - r$ is even. The consideration for dividing the analysis into two cases arises from the fact that the final index i involved in the vertex and edge labeling processes differs depending on whether $m - r$ is odd or even. Using the same approach as in determining the weight of the entire graph, we then obtain the corresponding subgraph weights for both cases—namely, when $m - r$ is odd and when $m - r$ is even—as follows

$$w\left(C_m^{(j)}\right) = 2(nm^2 - rnm + mr) + (r-1)n + m - r + j,$$
$$\text{for } 1 \le j \le n.$$

This shows that the weights of the subgraphs $C_m^{(j)}$ for $1 \le j \le n$form an arithmetic sequence with common difference 1. Consequently, the generalized book graph $B_{r,m}^{(n)}$ is a super $(a,1) - C_m$ −antimagic total graph with $a = 2(nm^2 - rnm + mr) + (r-1)n + m - r$. Hence, we obtain the following theorem, whose proof follows directly from the arguments presented above.

Theorem 2.4 For any $n, r \ge 2$, $m \ge 3$, and $r < 3$, the generalized book graph $B_{r,m}^{(n)}$ is a super $(a,1) - C\_m$ −antimagic total graph.

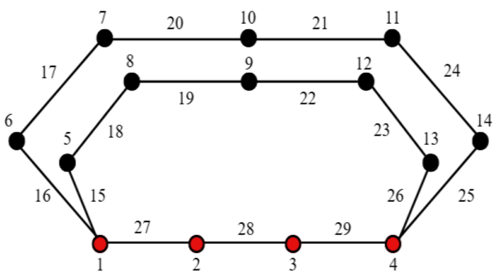

Figure 4. The super (64,1)-antimagic total labeling of the generalized book graph $B_{4,9}^{(2)}$

Finally, Figure 4 provides the super (64,1)-antimagic total labeling of the generalized book graph $B_{4,9}^{(2)}$

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