International Journal of Research and Review
[Indexed, Peer-Reviewed Journal]
Volume 12; Issue: 6; June 2025
Website: www.ijrrjournal.com
Research Paper *E-ISSN: 2349-9788; P-ISSN: 2454-2237*

Bifurcation and Dynamics in a 3D System with Quadratic Coupling

Nurcahya Yulian Ashar[1], Hafidh Khoerul Fata[1]

[1]Department of Mathematics, Faculty of Sciences and Mathematics, Diponegoro University, Semarang, Indonesia.

Corresponding Author: Hafidh Khoerul Fata

DOI: https://doi.org/10.52403/ijrr.20250676

ABSTRACT

This research explores the behavior of a three-dimensional autonomous nonlinear system featuring both quadratic and bilinear interaction terms. The system is developed as an extension of the SE8 model proposed by Molaie et al., with modifications that introduce additional nonlinearities through the inclusion of x^2 and y^2 components in the third equation. Although the system has a single equilibrium point and a relatively simple structure, it displays a wide range of complex dynamical phenomena, such as bifurcations, multistability, and chaotic attractors. To analyze these dynamics, equilibrium points are identified and their local stability is examined using Jacobian matrix linearization, revealing how changes in parameter values can lead to different stability outcomes. A thorough bifurcation study is conducted, focusing especially on the emergence of Hopf bifurcations. The theoretical results are validated through numerical simulations, which include bifurcation diagrams, Lyapunov exponent analysis, and phase portraits that depict the system's transition to chaos and its sensitivity to initial conditions.

Keywords: Nonlinear Dynamical Systems, Quadratic Coupling, Bifurcation Analysis, Chaos Theory, Lyapunov Exponents.

INTRODUCTION

Research on nonlinear dynamical systems has played a fundamental role in revealing the intricacies of time-dependent behaviors found in disciplines such as physics [1][2][3], engineering [4][5][6], biology [7][8][9], and economics [10][11][12]. In particular, low-dimensional autonomous systems—especially those incorporating quadratic or bilinear nonlinearities—have proven to be highly effective frameworks for investigating dynamic phenomena like bifurcations, chaotic motion, and multistable states. Although these systems are relatively simple from a mathematical standpoint, they are capable of producing a wide variety of complex behaviors, including periodic oscillations, strange attractors, and shifts between different stability conditions.

Quadratic nonlinear systems have garnered significant interest due to their capacity to capture complex dynamical behavior while still allowing for analytical treatment. Recent studies have shown that even the simplest quadratic models can produce elaborate bifurcation patterns and chaotic dynamics. For example, [13] found that a three-dimensional system composed solely of quadratic terms can give rise to diverse chaotic behaviors, including multistability and extreme sensitivity to initial conditions. In a related study, [14] explored a discrete version of a quadratic map and identified the presence of hyperchaotic attractors—characterized by three positive Lyapunov exponents—further emphasizing the depth and richness of dynamics that such systems can exhibit.

In this study, we examine an extended version of the SE8 system, which was initially introduced by Molaie, Jafari, Sprott, and Hashemi Golpayegani [15] is recognized for exhibiting chaotic dynamics even with only a single equilibrium point. The modified system builds upon the original SE8 framework by adding further quadratic coupling terms into the third differential equation, notably through the inclusion of both x^2 and y^2 components. The governing equations of the model are formulated as follows:

$$\dot{x} = y$$
$$\dot{y} = -x + yz$$
$$\dot{z} = -z + ax^2 + y^2 + b$$

where $x(t), y(t), z(t) \in \mathbb{R}$ represent the system's state variables as functions of time, and $a, b \in \mathbb{R}$ are real-valued parameters that govern the intensity of the system's internal nonlinear interactions and the magnitude of external forcing, respectively.

The system retains key characteristics commonly found in dissipative nonlinear systems: it is autonomous, incorporates energy input through self-excitation terms $ax^2 + y^2$, and exhibits energy dissipation through the linear damping term $-z$. The presence of the bilinear term yz in the second equation introduces a nonlinear feedback loop that significantly influences the system's stability and its long-term dynamical behavior. Despite its structural simplicity, the system displays complex dynamics, making it an effective model for investigating bifurcation processes and the onset of chaotic behavior.

This study is guided by three main objectives. First, we seek to identify the equilibrium points of the nonlinear dynamical system under consideration and evaluate their local stability by applying Jacobian matrix linearization. This step is crucial for understanding the system's foundational structure and the behavior of small perturbations in the vicinity of equilibrium. Second, we analyze how variations in the parameters a and b influence the system's qualitative dynamics, with a focus on detecting and analyzing Hopf bifurcations, which often mark the transition to oscillatory or chaotic motion. Third, we conduct extensive numerical simulations to examine the emergence of complex attractors and chaotic dynamics. These analyses are reinforced through the construction of bifurcation diagrams and the calculation of Lyapunov exponents, offering a thorough perspective on the system's long-term behavior and its sensitivity to initial conditions.

MATHEMATICAL MODEL

We examine a three-dimensional nonlinear dynamical system characterized by quadratic interactions as follows:

$$\dot{x} = y$$
$$\dot{y} = -x + yz$$
$$\dot{z} = -z - 0.7x^2 + y^2 - a$$

Where $x(t), y(t), z(t) \in \mathbb{R}$ represents the system states as a function of time , and $a \in \mathbb{R}$ are real-valued parameters that govern the intensity of nonlinear feedback and external forcing, respectively.

This system represents a generalized version of the SE8 system, initially introduced by Molaie, Jafari, Sprott, and Hashemi Golpayegani in their classification of simple chaotic flows characterized by a single stable equilibrium (1). The generalization entails extending the quadratic structure in the third equation to encompass both x^2 and y^2, while maintaining the essential nonlinear and dissipative properties that facilitate complex dynamical behaviour, including bifurcations and chaos.

The system exhibits several notable features. It is characterized by autonomy and nonlinearity, incorporating both quadratic and bilinear terms. The first equation is linear, serving as a bridge between x and y. The second equation introduces a bilinear interaction term yz, which contributes to nonlinearity through state-dependent feedback. The third equation integrates damping trough the linear term z, external forcing through the constant a, and self-

excitation via the quadratic terms $-0.7x^2$ and y^2.
The parameters a play essential role in determining the system's qualitative behavior. By varying these parameters, we will explore the equilibrium structure, stability, and bifurcation phenomena of the system in subsequent sections.

EQUILIBRIUM POINTS AND LOCAL STABILITY

To comprehend the qualitative behaviour of a dynamic system, it is essential to first identify its equilibrium points and assess their local stability. In this section, we systematically ascertain the equilibrium points of the system and evaluate their stability through linearization and eigenvalue analysis.

Determine of Equilibrium Points

The equilibrium points were obtained by setting the right-hand side of the system to zero.

$$\begin{cases} \dot{x} = y = 0 \\ \dot{y} = -x + yz = 0 \\ \dot{z} = -z - 0.7x^2 + y^2 - a \end{cases}$$

From the first equation, we have

$$y = 0$$

Substituting this into the second equation yields

$$-x = 0 \Rightarrow x = 0$$

Substituting $x = 0$ and $y = 0$ into the third equation gives:

$$-z - 0.7(0)^2 + (0)^2 - b = 0 \Rightarrow z = -a$$

Hence, the system has a unique equilibrium point at

$$E = (0,0,-a)$$

This single equilibrium is structurally simple, yet dynamically rich, as shown through stability and bifurcation analysis.

Linearization via the Jacobian Matrix

To study the local behaviour near the equilibrium, we linearize the system using the Jacobian matrix. The Jacobian of the system is

$$J(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & z & y \\ -1.4x & 2y & -1 \end{pmatrix}$$

Evaluating this at the equilibrium point $E = (0,0,b)$ we obtain

$$J(0,0,-a) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -a & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This is a block-triangular matrix that allows us to easily find the eigenvalues by computing the characteristic polynomial:

$$det(J - \lambda I) = (\lambda + 1) \cdot \begin{vmatrix} -\lambda & 1 \\ -1 & b - \lambda \end{vmatrix}$$
$$= (\lambda + 1)(\lambda^2 + a\lambda + 1)$$

Thus, the eigenvalues are

$$\lambda_1 = -1, \quad \lambda_{2,3} = \frac{-a \pm \sqrt{a^2 - 4}}{2}.$$

Stability Classification

The nature of the eigenvalues $\lambda_{2,3}$ changes with the value of the parameter a, leading to different types of stability at the equilibrium point. The classifications are as follows:

Case I: $a < -2$

For $a < -2$ the discriminant $a^2 - 4 > 0$, resulting in two real and distinct eigenvalues. Since $a < 0$, it follows that $-a > 0$, hence both λ_2 and λ_3 are real positive. With $\lambda_1 = -1$ and two positive eigenvalues, the equilibrium point is a saddle focus, and the system is unstable due to the presence of divergent trajectories.

Case II: $-2 < a < 2$

For $-2 < a < 2$, the discriminant $a^2 - 4 < 0$, and the eigenvalues $\lambda_{2,3}$ are complex conjugates with real part $-\frac{a}{2}$. The stability thus depends on the sign of a. If $0 < a < 2$ then $Re(\lambda_{2,3}) < 0$, thus the equilibrium is a stable spiral sink. If $-2 < a < 0$, then $Re(\lambda_{2,3}) > 0$, thus the equilibrium is an unstable spiral source. If $a = 0$ then $\lambda_{2,3} = \pm i$ thus the equilibrium is a center and may lead to a Hopf Bifurcation. This intermediate region $(-2 < a < 2)$ represents a critical transition zone where the system may undergo qualitative changes in stability and where oscillatory dynamics are most prominent.

***Case III*: $\alpha > 2$**

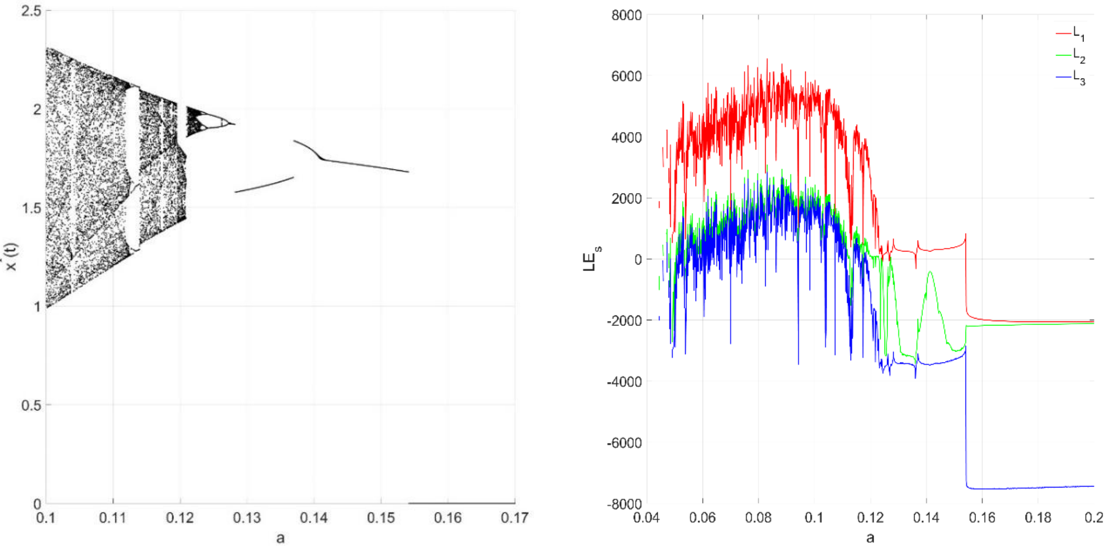
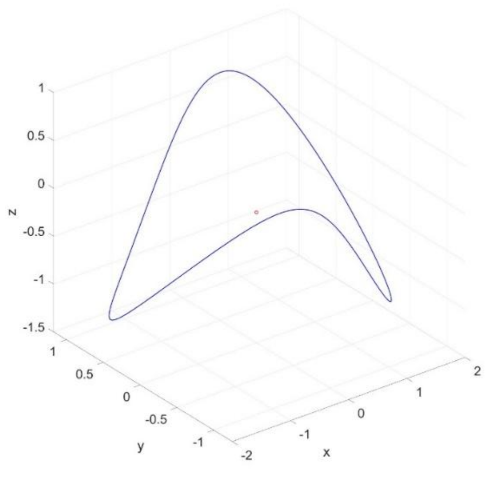
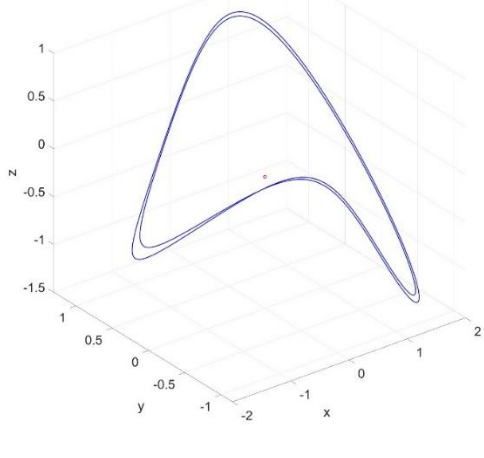


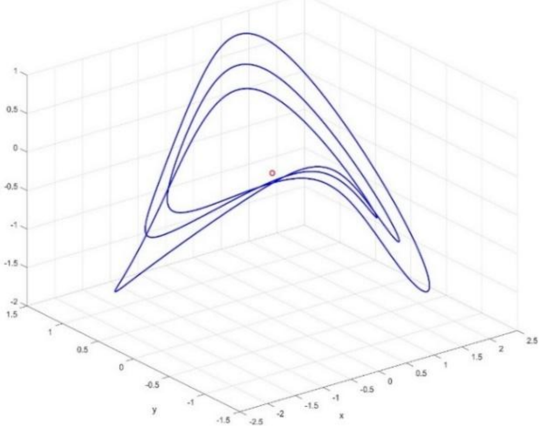
Figure 3.1 Global bifurcation and Lyapunov Exponents diagrams of the system (1) which is period-doubling cascades for α ∈ [0.1,0.17]



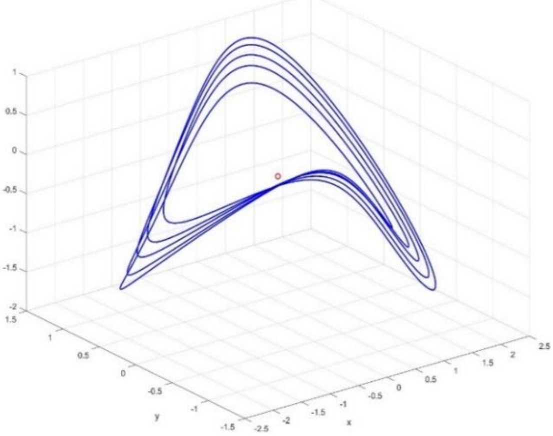
(a) α =0.165787



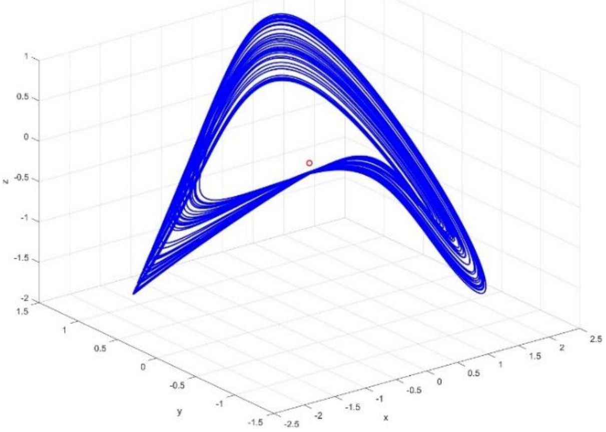
(b) α =0.125225



(c) α =0.113664



(d) α =0.117307



(e) α =0.110200

Figure 3.3 Phase portraits of the system for various values of the parameter α.

DISCUSSION AND CONCLUSION

This study has explored the bifurcation structure and dynamical complexity of a modified SE8 system, characterized by quadratic nonlinearities and a single equilibrium point. By extending the original SE8 model to include both $x^2$and y^2terms in the third equation, the system exhibits enhanced nonlinear coupling and a broader range of dynamical phenomena. Analytical investigation revealed that the system always admits a unique equilibrium point at $E = (0,0,-a)$, whose stability critically depends on the parameter aaa. Through linearization and eigenvalue analysis, we identified three distinct stability regimes: a saddle focus for$a < -2$, spiral behavior (sink or source) for $-2 < a < 2$, and an unstable node for $a > 2$, with a potential Hopf bifurcation occurring near $a = 0$.

Numerical simulations supported these theoretical predictions. Bifurcation diagrams and Lyapunov exponent calculations confirmed the onset of chaos through period-doubling cascades as the control parameter aaa varied within the range$[0,0.17]$. Phase portraits further illustrated the emergence of strange attractors, highlighting the system's sensitivity to initial conditions and parameter changes. The presence of positive Lyapunov exponents verified chaotic behavior for specific values of aaa, particularly in the vicinity of ≈ 0.117307, where the attractor becomes fully developed and structurally complex.

Overall, the results demonstrate that the generalized SE8 system provides a compact yet powerful framework for studying nonlinear phenomena such as bifurcations and chaos. Its simple structure makes it an attractive candidate for theoretical analysis and numerical exploration, while its rich dynamics position it as a potential benchmark system for future research on low-dimensional chaos.

Declaration by Authors

Acknowledgement: This study was supported by the Faculty of Science and Mathematics, Diponegoro University, through funding from the Dana Selain Undip number 26.G/UN7.F8/PP/II/2025.

Source of Funding: None

Conflict of Interest: No conflicts of interest declared.

REFERENCES

1. Wang, W.-X., Lai, Y.-C., and Grebogi, C. 2016. Data based identification and prediction of nonlinear and complex dynamical systems. Physics Reports, 644, 1–76.
2. Vincent, U. E., and Kolebaje, O. 2020. Introduction to the dynamics of driven nonlinear systems. Contemporary Physics, 61(3), 169–192.
3. Wang, B., et al. 2024. Automatic identification of dynamical system excited by time-dependent factor without prior information. Nonlinear Dynamics, 112(5), 3441–3452.

4. Zhang, Z., Shen, Q., and Wang, X. 2023. Parameter identification framework of nonlinear dynamical systems with Markovian switching. Chaos: An Interdisciplinary Journal of Nonlinear Science, 33(12).
5. Ghosh, D., et al. 2024. Recent achievements in nonlinear dynamics, synchronization, and networks. Chaos: An Interdisciplinary Journal of Nonlinear Science, 34(10).
6. Amjad, E., and Sokouti, B. 2025. Kinetics and dynamics of biological systems. In Systems Biology and In-Depth Applications for Unlocking Diseases, Academic Press, 59–67.
7. Dhakshinamoorthy, V., Wu, G.-C., and Banerjee, S. 2024. Chaotic Dynamics of Fractional Discrete Time Systems. CRC Press.
8. Martínez, J. H., Lehnertz, K., and Rubido, N. 2025. Introduction to Focus Issue: Data-driven models and analysis of complex systems. Chaos: An Interdisciplinary Journal of Nonlinear Science, 35(3), 030401.
9. Zhang, W.-B. 2013. Synergetic economics: time and change in nonlinear economics, Vol. 53. Springer Science & Business Media.
10. Mariono, M., et al. 2024. Forecasting analysis with the dynamic systems approach on economic data. In Proceeding of International Seminar On Student Research In Education, Science, and Technology, Vol. 1.
11. Sarkar, B., Bhattacharya, S., and Bairagi, N. 2023. Dynamic behaviour of a single-species nonlinear fishery model with infection: the role of fishing tax and time-dependent market price. Journal of Nonlinear Sciences & Applications (JNSA), 16(3).
12. Al-Taani, H., et al. 2024. Asymmetry and symmetry in new three-dimensional chaotic map with commensurate and incommensurate fractional orders. Symmetry, 16(11), 1447.
13. Muni, S. S. 2024. Pathways to hyperchaos in a three-dimensional quadratic map. The European Physical Journal Plus, 139(7), 636.
14. Shen, C., Chen, M., Zhang, Z., and Lü, J. (2014). Designing hyperchaotic systems with any desired number of positive Lyapunov exponents via a simple model. IEEE Transactions on Circuits and Systems I: Regular Papers, 61(8), 2380–2389.
15. Jafari, S., Sprott, J. C., and Hashemi Golpayegani, S. M. R. 2013. Elementary quadratic chaotic flows with no equilibria. Phys. Lett. A, 377(9), 699–702. Available from: http://dx.doi.org/10.1016/j.physleta.2013.01.009.

How to cite this article: Nurcahya Yulian Ashar, Hafidh Khoerul Fata. Bifurcation and dynamics in a 3D system with quadratic coupling. *International Journal of Research and Review*. 2025; 12(6): 669-674. DOI: *https://doi.org/10.52403/ijrr.20250676*

******