International Journal of Research and Review
[Indexed, Peer-Reviewed Journal]
Volume 13; Issue: 6; June 2026
Website: www.ijrrjournal.com
Research Paper *E-ISSN: 2349-9788; P-ISSN: 2454-2237*

Application of Kushare Transform in Fractional Differential Equation

Savita S. Khakale[1], Kailas S. Ahire[2]

[1]Research Scholar, M.S, G. Arts, Commerce and Science College, Malegaon Camp Malegaon 423203,
[2]Head Department of Mathematics, M.S, G. Arts, Commerce and Science College, Malegaon Camp Malegaon 423203

Corresponding Author: Savita S. Khakale

DOI: https://doi.org/10.52403/ijrr.20260641

ABSTRACT

Nowadays, integral transforms are widely recognized as important techniques for solving differential equations. In this work, the Kushare transform is applied to obtain solutions of linear homogeneous and non-homogeneous fractional differential equations with constant coefficients. The fractional derivatives are defined in the Caputo sense. Moreover, the Kushare transform serves as an efficient and versatile method for dealing with different forms of fractional differential equations.

Keywords: Kushare Transform, Fractional Differential Equations, Integral transforms.

1. INTRODUCTION

Fractional differential calculus is an important area of mathematics with wide-ranging applications. Fractional calculus relates to a classical standard result from differential and integral Calculus. Kushare transform is described in various fields. Integral transforms have been extensively applied in multiple fields, including engineering, control engineering, mechanics, science, bioengineering, image processing, and viscoelasticity. The integral transform was first developed by Pierre-Simmon Laplace in 1780. Since then, many researchers have established a lot of integral transformations of Laplace types like Laplace, Sumudu, Elazaki, Kamal, Sawi, Snehu, Mahgoub, and Aboodh transforms. In recent days, Patil [1, 2] applied the Sawi transform for the Bessels function and the error function.

Integral transforms are widely regarded as effective mathematical methods for obtaining analytical solution of fractional differential equations. Laplace, Mellin, Sumudu, Elzaki, Kamal, Sawi, Snehu, Mahgoub, and Aboodh transforms are commonly employed as important mathematical techniques for obtaining solutions of fractional differential equations [6]. Sachin R. Kushare, D. P. Patil, and A. M. Takate introduced the Kushare integral transform [3] to solve ordinary differential equations. Lately, D. P. Patil and S. L. Kandalkar, N. D. Gatkal, [4] used the Kushare Transform to solve the system of ordinary differential equations of the first order and first degree. D.P. Patil, P. S. Nikam, and P. D. Shinde [5] used the Kushare transform to obtain the solution of the linear Volterra integral equation of the first kind. Fractional differential equations and methods of solution were well explained by I. Podlubny [6] in 1988. Puhpam A. E., Lydia S. K. [7] introduced the Mahgoub Transform Method for Solving Linear Fractional Differential

Equations. Aruldoss R., Devi R. A. [8] introduced used the Aboodh transform to solve fractional differential equations.
Section 2nd includes some basic definitions and related methods to fractional calculus. In sections 4th and 5th, we illustrate the application of the Kushare transform method in solving fractional differential equations to derive exact solutions involving Caputo derivatives.

2. MATERIAL & METHODS

2.1 Basic Definitions:

In this section, we introduce some fundamental definitions and properties of fractional calculus and Soham transform.

Definition 1. Caputo Fractional Derivative
For $\alpha\epsilon\mathbb{R}, \alpha > 0,$ then Caputo fractional derivative of order $\alpha\, of\, f(x)$ is denoted by ${}^{C}D^{\alpha}\, f(x)$ and is defined by

$$^{C}D^{\alpha}\, f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^m(t)dt\ , \qquad x > 0 \qquad (1)$$

Where $m-1 \leq \alpha \leq m, m \in N^+\ and\ \Gamma(.)$ denotes the Gamma function.

Definition 2. Reimann-Liouville Fractional Integral

For $f(x) \in C_\mu, \mu \geq -1$ then the Reimann Liouville fractional integral I of $f(x)$ of order $\alpha \in \mathbb{R}, \alpha > 0$ is denoted by $I^\alpha f(x)$ and is defined as

$$I_x{}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt\ , \qquad (2)$$

Where $x > 0\ and\ I^0 f(x) = f(x)$

Definition 3. Mittag-Leffler function of one parameter
For parameters $\alpha, \beta > 0$ the Mittag-Leffler function of one parameter α is denoted by $E_\alpha(x)$ and is defined as

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}\ , \qquad \alpha > 0, x \in C \qquad (3)$$

and $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function with two parameters $\alpha\ and \beta$ is defined as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}\ , \qquad \alpha > 0, Re(\alpha), Re(\beta) > 0 \qquad (4)$$

Where C is the set of complex numbers and the direct generalization of exponential series with $\beta = 1,$ we get, $E_{\alpha,1}(x) = E_\alpha(x)$.

Definition 4. Kushare Transform
The Kushare transform is defined as a function of exponential order. We consider functions in the set B defined by

$$B = \{f(t): \exists M, k_1, k_2 > 0, |f(t)| < Me^{|t|k_j}, if\ t \in (-1)^j \times [0,\infty)\} \qquad (5)$$

For a given function in the set A, the constant M must be a finite number, k_1, k_2 may be finite or infinite.
Kushare transform denoted by the operator $S(v)$ defined by the integral equations

$$K[f(t)] = S(v) = v\int_0^\infty f(t)e^{-v^\alpha t}dt \qquad (6)$$

Where α is nonzero real numbers $t \geq 0, \quad k_1 \leq v \leq k_2$

Table. Kushare Transform of Some Functions as Follows Kushare Transform of Some Functions

Sr. No.	F(x)	$\delta\{f(x)\} = P(v)$
1	1	$\frac{1}{v^{\alpha-1}}$
2	x	$\frac{1}{v^{2\alpha-1}}$
3	x^2	$\frac{2}{v^{3\alpha-1}}$
4	x^3	$\frac{6}{v^{4\alpha-1}}$
5	$x^m, m \geq 0$	$\frac{\Gamma(m+1)}{v^{m\alpha+\alpha-1}}$
6	e^{ax}	$\frac{v}{(v^{\alpha}-a)}$
7	$\sin(ax)$	$\frac{av}{(v^{2\alpha}+a^2)}$
8	$\cos(ax)$	$\frac{v^{\alpha+1}}{(v^{2\alpha}+a^2)}$

2.2 Kushare Transform for Derivatives of Function $f(x)$

Let $K\{f(x)\} = S(v)$ are:

(i) $K\{f'(x)\} = v^{\alpha}S(v) - vf(0)$

(ii) $K\{f''^{(x)}\} = v^{2\alpha}S(v) - v^{\alpha+1}f(0) - vf'(0)$

(iii) $K\{f^{n}(x)\} = v^{n\alpha}S(v) - v\sum_{k=0}^{n-1} v^{\alpha(n-k-1)}f^{k}(0)$

2.3 Inverse Theorem for Kushare Transform

Inverse Kushare transform of $f(t)$ is $S(v)$ and K^{-1} is called the inverse Kushare transform operator then inverse Kushare transform is defined as

$$K^{-1}[S(v)] = f(t) \tag{10}$$

2.4 Lemma 1. Convolution Theorem for Laplace Transform

. If $\ell\{f(t)\} = f(s)$ and ℓ{g(t)}=g(s), then

$$\ell\{f(t) * g(t)\} = \ell\{t(t)\}\ell\{g(t)\} = f(s)g(s)$$

2.5 Lemma 2. Convolution Theorem for Kushare Transform

Let $f(t)$ and $g(t)$ be two functions and $K\{f(t)\} = F(v)$and $K\{g(t)\} = G(v)$. The convolution of f and g denoted by $f * g$ is the function on $u \geq 0$ given by

$$f * g(t) = \int_{u=0}^{x} f(u)g(t-u)du \;\; then\; K[f * g(t)] = vK[f(t)]K[g(t)]$$
$$= vF(v)G(v) \tag{11}$$

3. RESULTS

3.1 Theorem 1. For $m-1 < \alpha \leq m, m \in N$ and if $S(v)$ is Kushare transform of $f(t)$, then Kushare transform of Riemann-Liouville fractional integral of order α is

$$K[I^{\alpha}f(t)] = \frac{1}{v^{n\alpha}}S(v) \tag{12}$$

Proof:- Suppose the Riemann-Liouville fractional integral of function $f(x)$

$$I_t^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau$$

$$I_t^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}[t^{\alpha-1} * f(t)]$$

Taking Kushare transform to the above equation on both sides, we get,

$$K[\ \ I_t{}^{\alpha} f(t)] = K\left\{\frac{1}{\Gamma(\alpha)}[t^{\alpha-1} * f(t)]\right\}$$

$$= \frac{1}{\Gamma(\alpha)}\ K[t^{\alpha-1} * f(t)]$$

By Lemma (2)

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{v}\ K[t^{\alpha-1}]\ K[f(t)]$$

$$= \frac{1}{\Gamma(\alpha)}\ \frac{1}{v}\frac{\Gamma(\alpha-1+1)}{v^{n\alpha-n+n-1}}\ S(v)$$

$$\delta[I^{\alpha} f(x)] = \frac{1}{v^{n\alpha}} S(v)$$

3.2 Theorem 2. For $m-1<\alpha\leq m, m\in N$ and if $S(v)$ is Kushare transform of $f(x)$, then Kushare transform of Caputo α fractional derivative of function $f(x)$ is

$$K\left[\,{}^{C}D^{\alpha}\ f(x)\right] = \frac{1}{v^{n(m-\alpha)}}\left\{v^{mn}S(v) - v\sum_{k=0}^{m-1} v^{n(m-1-k)}\ f^{k}(0)\right\} \tag{13}$$

Proof:- Suppose the Caputo fractional derivative of function $f(x)$ is

$${}^{C}D^{\alpha}\ f(x) = I^{m-\alpha}D^{m}(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-\tau)^{m-\alpha-1} f^{m}(\tau)d\tau$$

Taking the Kushare transform of the above equation on both sides, we get,

$$K\left[\,{}^{C}D^{\alpha}\ f(x)\right] = \frac{1}{\Gamma(m-\alpha)}\ K[x^{m-\alpha-1} * f^{m}(x)]$$

$$= \frac{1}{\Gamma(m-\alpha)}\ \frac{1}{v}\ K[x^{m-\alpha-1}] * K[f^{m}(x)]$$

By Lemma (2)

$$= \frac{1}{\Gamma(m-\alpha)}\frac{1}{v}\ \frac{\Gamma(m-\alpha-1+1)}{v^{n(m-\alpha-1)+n-1}}\left\{v^{nm}S(v) - v\sum_{k=0}^{m-1} v^{n(m-1-k)} f^{k}(0)\right\}$$

$$= \frac{1}{\Gamma(m-\alpha)v}\frac{\Gamma(m-\alpha)}{v^{nm-n\alpha-n+n-1}}\ \left\{v^{nm}S(v) - v\sum_{k=0}^{m-1} v^{n(m-1-k)} f^{k}(0)\right\}$$

$$= \frac{1}{v^{n(m-\alpha)}}\ \left\{v^{nm}S(v) - v\sum_{k=0}^{m-1} v^{n(m-1-k)} f^{k}(0)\right\}$$

$$\therefore K\left[\,{}^{C}D^{\alpha}\ f(x)\right] = \ \frac{1}{v^{n(m-\alpha)}}\left\{v^{nm}S(v) - v\sum_{k=0}^{m-1} v^{n(m-1-k)} f^{k}(0)\right\}$$

3.3 Theorem 3. If $\alpha,\beta>0$ and the Mittag- Leffler function with two parameters α and β is $E_{\alpha,\beta}(.)$ then we have inverse Kushare transform formula is given by

$$K\left[t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\right] = \quad \frac{v^{n\alpha}}{v^{n\beta+1}(v^{n\alpha}-a)} \tag{14}$$

Proof:- By using the definition of Kushare transform of the function $t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$, we have

$$K\left[t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\right] = v\int_0^{\infty} t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})e^{-v^{n\alpha}t}\ dt$$

$$= \sum_{k=0}^{\infty}\frac{a^{k}t^{\alpha k}}{\Gamma(\alpha k+\beta)} v\ \int_0^{\infty} t^{\beta-1}e^{-v^{n}t}dt$$

$$= \sum_{k=0}^{\infty}\frac{a^{k}}{\Gamma(\alpha k+\beta)} v\ \int_0^{\infty} t^{\alpha k+\beta-1}e^{-v^{n}t}dt$$

$$= \sum_{k=0}^{\infty}\frac{a^{k}}{\Gamma(\alpha k+\beta)}\ K\left[t^{\alpha k+\beta-1}\right]$$

$$= \sum_{k=0}^{\infty}\frac{a^{k}}{\Gamma(\alpha k+\beta)}\ \frac{\Gamma(\alpha k+\beta)}{v^{n(\alpha k+\beta-1+1)-1}}$$

$$= \sum_{k=0}^{\infty}\frac{a^{k}}{v^{n\alpha k+n\beta-1}}$$

$$= \frac{1}{v^{n\beta-1}}\sum_{k=0}^{\infty}\left(\frac{a}{v^{n\alpha}}\right)^{k} \qquad \left(\because \sum_{k=0}^{\infty} a^{k} = \frac{1}{1-a}\right)$$

$$= \frac{1}{v^{n\beta-1}}\frac{1}{1-\frac{a}{v^{n\alpha}}}$$

$$= \frac{v^{n\alpha}}{v^{n\beta-1}(v^{n\alpha}-a)}$$

$$\therefore \delta\left[x^{\beta-1}E_{\alpha,\beta}(ax^{\alpha})\right] = \frac{v^{n\alpha}}{v^{n\beta+1}(v^{n\alpha}-a)}$$

Taking inverse Kushare transform on both sides, we get,

$$x^{\beta-1}E_{\alpha,\beta}(ax^{\alpha}) = \delta^{-1}\left[\frac{v^{n\alpha}}{v^{n\beta-1}(v^{n\alpha}-a)}\right]$$

3.4 Applications

In this section, we establish the Kushare transform method for solving some fractional differential equations to obtain exact solutions with Caputo derivatives.

Example 1. Consider the following linear inhomogeneous equation

$$^{C}D^{\alpha}y(x) + y(x) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x \tag{15}$$

Subject to the initial condition $y(0) = 0,$ where $0 \leq \alpha \leq 1.$

Solution: Applying the Kushare transform to the above equation (15) on both sides, we get,

$$K[\,^{C}D^{\alpha}y(x) + y(x)] = K\left[\frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}\right] - K\left[\frac{x^{1-\alpha}}{\Gamma(2-\alpha)}\right] + K[x^2] - K[x]$$

By using Lemma (2) to the above equation and using condition we get,

$$\frac{1}{v^{n(m-\alpha)}}\{v^{nm}S(v) - v\,v^{nm-1}y(0)\} + S(v) = \frac{2}{v^{3n-n\alpha-1}} - \frac{1}{v^{2n-n\alpha-1}} + \frac{\Gamma(3)}{v^{3n-1}} - \frac{1}{v^{2n-1}}$$

$$(v^{n\alpha}S(v) + S(v)) = \frac{2}{v^{3n-n\alpha-1}} + \frac{\Gamma(3)}{v^{3n-1}} - \frac{1}{v^{2n-n\alpha-1}} - \frac{1}{v^{2n-1}}$$

$$(v^{n\alpha}+1)S(v) = 2\left(\frac{1}{v^{3n-n\alpha-1}} + \frac{1}{v^{3n-1}}\right) - \left(\frac{1}{v^{2n-n\alpha-1}} + \frac{1}{v^{2n-1}}\right) \quad (\because(3) = 2)$$

$$= 2\left(\frac{v^{n\alpha}}{v^{3n-1}} + \frac{1}{v^{3n-1}}\right) - \left(\frac{v^{n\alpha}}{v^{2n-1}} + \frac{1}{v^{2n-1}}\right)$$

$$(v^{n\alpha}+1)S(v) = 2\left(\frac{v^{n\alpha}+1}{v^{3n+1}}\right) - \left(\frac{v^{n\alpha}+1}{v^{2n+1}}\right)$$

$$S(v) = \frac{2}{v^{3n+1}} - \frac{1}{v^{2n+1}}$$

Applying the inverse of the Kushare transform, we get,

$$K^{-1}[S(v)] = K^{-1}\left[\frac{2}{v^{3n+1}}\right] - K^{-1}\left[\frac{1}{v^{2n+1}}\right]$$

$$y(x) = x^2 - x$$

Example 2. Consider the initial value problem of the non-homogeneous Bagley-Torvik equation

$$D^2y(x) + \,^{C}D^{\frac{3}{2}}y(x) + y(x) = 1 + x \tag{16}$$

Subject to the initial conditions $y(0) = y'(0) = 1.$

Solution: Applying Kushare transform of equation (16) on both sides, we get,

$$K[D^2y(x)] + K\left[\,^{C}D^{\frac{3}{2}}y(x)\right] + K[y(x)] = K[1 + x]$$

Using theorem (2) we get,

$$\left(v^{2n}S(v) - v^{n+1}y(0) - vy'(0)\right) + v^{\frac{3}{2}\beta}S(v) - \frac{v^{\frac{3}{2}\beta}}{v^{\beta m+1}}\left(v^{\beta(m-1)}y(0) + v^{\beta(m-2)}y'(0)\right) + P(v) = \frac{1}{v^{\beta+1}} + \frac{1}{v^{2\beta+1}}$$

$$\left(v^{2\beta} + v^{\frac{3}{2}\beta} + 1\right)P(v) - \left(v^{\beta-1} + \frac{1}{v} + \frac{v^{\left(\frac{3}{2}\right)\beta}}{v^{\beta+1}} + \frac{v^{\left(\frac{3}{2}\right)\beta}}{v^{2\beta+2}}\right) = \frac{1}{v^{\beta+1}} + \frac{1}{v^{2\beta+1}}$$

$$\left(v^{2\beta}+v^{\frac{3}{2}\beta}+1\right)P(v)=\left(\frac{1}{v^{\beta+1}}+v^{\beta-1}+\frac{v^{\left(\frac{3}{2}\right)\beta}}{v^{\beta+1}}\right)+\left(\frac{1}{v^{2\beta+1}}+\frac{1}{v}+\frac{v^{\left(\frac{3}{2}\right)\beta}}{v^{2\beta+2}}\right)$$

$$\therefore P(v)=\frac{1}{v^{\beta+1}}+\frac{1}{v^{2\beta+1}}$$

Taking the inverse Kushare transform of the above equation on both sides, we get,

$$K^{-1}[P(v)]=K^{-1}\left[\frac{1}{v^{\beta+1}}\right]-K^{-1}\left[\frac{1}{v^{2\beta+1}}\right]$$

we get, the analytical solution of the above equation is,

$$y(x)=1+x$$

4. DISCUSSION

Integral transforms are among the most useful techniques in mathematics, employed to find solutions to differential equations, partial differential equations, integro-differential equations, delay differential equations, and population growth. Integral transform methods are highly effective techniques in mathematics, often used to obtain the analytical solution of fractional differential equations. Many researchers have shown great interest in finding numerical solutions to both linear and non-linear fractional differential equations. The connection of the Kushare transform with the Mahgoub, Pourreza, and Elzaki transform goes much deeper. In this paper, we also discuss the Kushare transform of fractional integrals and derivatives and its application of the Kushare transform to find the solution of fractional differential equation with Caputo sense.

5. CONCLUSION

In this study, the Kushare transform method is employed to solve several linear homogeneous and non-homogeneous fractional differential equations with constant coefficients. Three theorems associated with the Kushare transform are established. Various examples are presented to illustrate the exact solutions of fractional differential equations. The results indicate that the Kushare transform method is an efficient and reliable approach for obtaining analytical solutions of fractional differential equations with constant coefficients.

Declaration by Authors

Acknowledgement: None

Source of Funding: None

Conflict of Interest: No conflicts of interest declared.

6. REFERENCES

1. D. P. Patil, Application of Sawi transform in Bessel functions, Aayushi International Interdisciplinary Research Journal, Special issue No 86, pp. 171-175.
2. D. P. Patil, Application of Sawi transform of error function for evaluating improper integrals, Journal of Research and Development, 2021, Vol. 11, Issue 20, pp. 41-45.
3. Sachin Ramdas Kushare, D. P. Patil, Archana Madhukar Takate, “The New Integral Transform ‘KUSHARE Transform’,” International Journal of Advances in Engineering and Management (IJAEM), Vol. 3, Issue 9, Sep. 2021, pp. 1589–1592.
4. Dinkar P. Patil, Shweta L. Kandalkar, Nikita D. Gatkal, “Applications of ‘KUSHARE Transform’ in the System of Differential Equations,” International Advanced Research Jour-nal in Science, Engineering, and Technology, Vol. 9, Issue 7, July 2022. DOI: 10.17148/IAR-JSET.2022.9734.
5. Dinkar P. Patil, Poonam S. Nikam, Pragati D. Shinde, “Kushare transform in solving Faltung type Volterra integro-differential equation of the first kind,” International Advanced. Research Journal in Science, Engineering, and Technology, Vol. 9, Issue 10, October 2022.
6. Podlubny I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of

their solution and some of their applications. Elsevier, 1999.

7. Puhpam A. E., Lydia S. K., "Mahgoub Transform Method for Solving Linear Fractional. Differential Equations," International Journal of Mathematics Trends and Technology, Vol. 58, 2018, pp. 253–257.
8. Aruldoss R., Devi R. A., "Aboodh transform for solving fractional differential equations," Global Journal of Pure and Applied Mathematics, Vol. 16, 2020, pp. 145–153.

How to cite this article: Savita S. Khakale, Kailas S. Ahire. Application of Kushare transform in fractional differential equation. *International Journal of Research and Review*. 2026; 13(6): 419-425. DOI: *https://doi.org/10.52403/ijrr.20260641*

******